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Transport coefficients of strongly coupled gauge theories: Insights from string theory

A.O. Starinets[a]

Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo N2L 2Y5, Ontario, Canada

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Abstract. The transport properties of certain strongly coupled thermal gauge theories can be determined from their effective description in terms of gravity or superstring theory duals. Here we provide a short summary of the results for the shear and bulk viscosity, charge diffusion constant, and the speed of sound in supersymmetric strongly interacting plasmas. We also outline a general algorithm for computing transport coefficients in any gravity dual. The algorithm relates the transport coefficients to the coefficients in the quasinormal spectrum of five-dimensional black holes in asymptotically anti de Sitter space.

PACS. 11.25.Hf Conformal field theory, algebraic structures – 11.10.Wx Finite-temperature field theory

1 Introduction

The gauge theory —gravity or, more generally, gauge theory— string theory duality [1–3] (for reviews, see [4–6]) has been developed over the last eight years as a novel rigorous approach for understanding non-perturbative aspects of gauge theories. Together with lattice gauge theory, it provides a variety of useful insights into the dynamics of strongly coupled quantum systems. The major advantage of the gauge/gravity duality over the lattice methods is that it allows one to obtain precise analytical results in the regime of strong coupling. Its major disadvantage is that only a limited number of gauge theories, namely the ones for which gravity- or string theory duals are known, can be treated in this approach. Unfortunately, this class of theories does not yet include QCD, although there seems to be a consensus that a string theory dual to QCD should exist (see, *e.g.*, [7,8]). Even in the absence of a dual string theory description of QCD, results for other gauge theories and in particular thermal gauge theories, are of interest: first, they have a theoretical significance of their own; second, they may uncover universal properties of the strong-coupling dynamics shared by a large class of (if not all) gauge theories.

It is important to emphasize that the duality is a *conjecture* rather than an established mathematical or experimental fact. To date, it survived numerous theoretical tests and is generally believed to be true, although it is unlikely that a formal mathematical proof will ever be given. It is thus important to seek independent verifications of the results provided by the correspondence; in particular, a direct comparison with lattice gauge theory results would be highly desirable.

The original example of the gauge/gravity duality conjecture (also known as the AdS/CFT correspondence) was proposed in 1997 by J. Maldacena [1]. It is a statement that a certain (type IIB) string theory on $AdS_5 \times S^5$ (five-dimensional anti de Sitter space times a five-dimensional sphere) is *equivalent* to $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills (SYM) theory in four dimensions. The "extra" six dimensions of $AdS_5 \times S^5$ are interpreted as internal parameters of the Yang-Mills theory: the fifth dimension of the anti de Sitter space is related to the energy scale of the gauge theory, and the rotational group $SO(6)$ of the sphere is precisely the R-symmetry group of the $\mathcal{N} = 4$ SYM. The $\mathcal{N} = 4$ SYM is a superconformal theory with a vanishing beta-function. Thus, unlike in QCD, the coupling does not run and can be set to any desired value. The theory does not have any internal scale (such as Λ_{QCD}) and is not confining.

As stated, the conjectured equivalence is supposed to be valid for any range of the parameters of the gauge theory (*i.e.* for any N_c and any coupling). The limit of (infinitely) large N_c and (infinitely) large 't Hooft coupling $g^2_{YM} N_c$ in the gauge theory corresponds (on the string theory side of the duality) to a classical (super)gravity approximation to the full string theory. Most calculations in AdS/CFT are done in this limit. Corrections in powers of inverse 't Hooft coupling can be computed by considering stringy corrections to the classical anti de Sitter geometry. Corrections in inverse powers of N_c correspond to quantum gravity corrections. Finding these corrections is a difficult problem.

[a] e-mail: starina@perimeterinstitute.ca

Gravity duals of non-conformal theories can also be constructed (see, *e.g.*, [9,10]). These non-conformal theories are usually the mass deformations of the original $\mathcal{N}=4$ SYM theory (a "mass deformation" here means that the IR-relevant operators such as mass terms are added to the $\mathcal{N}=4$ SYM Lagrangian thus introducing a scale M into the theory). In the UV, *i.e.* for energies $E \gg M$, such theories reduce to $\mathcal{N}=4$ SYM. In the IR, they become $\mathcal{N}=2$, $\mathcal{N}=1$ SYM or a non-supersymmetric gauge theory —depending on the particular type of mass deformation. To emphasize the difference in the behavior of such theories above and below the mass deformation scale M, they are denoted $\mathcal{N}=2^*$, $\mathcal{N}=1^*$. These theories can exhibit confinement, chiral symmetry breaking and other interesting features —all of them identified and accessible to study through their effective gravity duals[1]. By shifting the mass deformation scale to the UV, one can —at least in principle— make these theories arbitrarily close to the "realistic" gauge theories (*e.g.*, $\mathcal{N}=1$ SYM). However, as the coupling becomes small in the UV, the dual classical gravity description breaks down and must be replaced by a full string theory description which is not understood at the moment.

Thermal properties of the gauge theories can be studied by including black holes in the (asymptotically) AdS background [1,11]. The Hawking temperature and the Bekenstein-Hawking entropy of the gravity background are identified, respectively, with the temperature and entropy of the gauge theory in thermal equilibrium. One of the earliest results in thermal AdS/CFT was the calculation of the entropy of $\mathcal{N}=4$ SYM in the regime of large N_c and large 't Hooft coupling [12].

In thermal gauge theories, of particular interest is the regime described by hydrodynamics. This near-equilibrium regime is completely characterized by transport coefficients (*e.g.*, shear and bulk viscosities) whose values are determined by the underlying microscopic theory. In practice they are hard to compute from "first principles", even in perturbation theory. (For example, no perturbative calculation of the bulk viscosity in a gauge theory seems to be available at the moment.) Lattice approaches to computing transport coefficients (see, *e.g.*, [13,14]) rely on indirect methods and to the best of our knowledge cannot yet provide unambiguous quantitative results. Thus, the AdS/CFT remains the only source of theoretical information about transport properties of thermal gauge theories in the non-perturbative regime, although the gauge theories in question do not include QCD.

In order to study the near-equilibrium processes needed for the calculation of transport properties in AdS/CFT, one considers perturbations of the black-hole background. These perturbations correspond to deviations from the thermal equilibrium in the dual gauge theory.

In this short review we provide a summary of the current state of affairs in computing the transport properties from gravity duals. More details can be found in the original articles [15–26].

[1] Quantitative study of these gravity duals may face formidable technical difficulties.

2 Methods for computing transport coefficients from dual gravity

Gauge/gravity duality allows one to compute the correlation functions of gauge-invariant operators from gravity. To deal with real-time processes in the gauge theory one needs the Lorentzian rather than Euclidean formulation of the correspondence. A concrete Lorentzian recipe for the two-point functions was given in [16] and generalized to higher-point functions in [27].

2.1 Viscosity from the Green-Kubo formula

Once the retarded correlation functions are known, transport coefficients can be obtained from Green-Kubo formulas. For example, for the shear viscosity we have

$$\eta = \lim_{\omega\to 0} \frac{1}{2\omega} \int \mathrm{d}t\, \mathrm{d}\boldsymbol{x}\, e^{i\omega t} \langle [T_{xy}(x),\, T_{xy}(0)]\rangle = - \lim_{\omega\to 0} \mathrm{Im}\, \frac{G^R_{xy,xy}(\omega,0)}{\omega}\,, \tag{1}$$

where the retarded Green's function for the components of the stress-energy tensor is defined as

$$G^R_{\mu\nu,\lambda\rho}(\omega,\boldsymbol{q}) = -i \int \mathrm{d}^4x\, e^{-iq\cdot x}\, \theta(t) \langle [T_{\mu\nu}(x),\, T_{\lambda\rho}(0)]\rangle\,. \tag{2}$$

Thus finding the shear viscosity is equivalent to computing the zero-frequency limit of the imaginary part of the retarded correlator $G^R_{xy,xy}$.

Such a computation can be carried out explicitly in the simplest example of thermal $\mathcal{N}=4$ $SU(N_c)$ SYM in the limit of infinite N_c and infinite 't Hooft coupling $g^2_{YM}N_c$. An equilibrium thermal state of the theory at temperature T is described by a dual five-dimensional AdS-Schwarzschild black hole (more precisely, by a black hole with a translationally invariant horizon) whose metric is given by

$$\mathrm{d}s_5^2 = \frac{(\pi T R)^2}{u} \left(-f(u)\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) + \frac{R^2}{4u^2 f(u)}\mathrm{d}u^2\,, \tag{3}$$

where $f(u) = 1 - u^2$, T is the Hawking temperature, R is the AdS radius. In eq. (3), the horizon corresponds to $u=1$, the spatial infinity to $u=0$. One may think of the four-dimensional gauge theory as of a theory defined on the "boundary" at $u\to 0$ of the spacetime (3) with the standard Minkowski coordinates t, x, y, z.

According to the Lorentzian version of the AdS/CFT, the retarded correlator $G^R_{xy,xy}$ is completely known (in the limit $g^2_{YM}N_c\to\infty$, $N_c\to\infty$) if the solution to the equation for a minimally coupled massless scalar propagating in the background (3) is known. The second-order differential equation should of course be supplemented by the appropriate boundary conditions, as discussed in [16]. Different boundary conditions at the (future) horizon correspond to different types of Green's functions (retarded, advanced, Feynman etc.) in the field theory. The (retarded) two-point function of the stress-energy tensor (in Fourier space) in the hydrodynamic approximation reads

$$G_{xy,xy}(\omega, \boldsymbol{q}) = -\frac{i\pi N_c^2 \omega T^3}{8} \,. \tag{4}$$

Using the Kubo formula (1) for the shear viscosity one obtains [15,17]

$$\eta = \frac{\pi}{8} N_c^2 T^3 \,. \tag{5}$$

Following 't Hooft's philosophy of the large-N_c expansion [28], the generic expression for the shear viscosity in conformal gauge theory can be written as

$$\eta = T^3 \sum_{k=0}^{\infty} N_c^{2-2k} \, f_k(g_{YM}^2 N_c) \,, \tag{6}$$

where the dependence on temperature follows from the dimensional analysis and the fact that for a thermal conformal theory the only scale is the temperature itself. The result (5) means that

$$\lim_{g_{YM}^2 N_c \to \infty} f_0 = \pi/8 \,. \tag{7}$$

Taking into account classical stringy corrections to the metric (3) (*i.e.*, the corrections arising from the fact that strings are one-dimensional rather than point-like objects) one can compute the correction to the result (5) [23]:

$$\eta = \frac{\pi}{8} N^2 T^3 \left(1 + \frac{75}{4} \zeta(3) \, (2 g_{YM}^2 N_c)^{-3/2} + \cdots \right) \,, \tag{8}$$

where $\zeta(3) \approx 1.2020569$ is Apéry's constant. Combining eq. (8) with the result for the strong-coupling limit of the volume entropy density [29]

$$s = S/V_3 = \frac{2\pi^2}{3} N_c^2 T^3 \left(\frac{3}{4} + \frac{45}{32} (2 g_{YM}^2 N_c)^{-3/2} + \cdots \right) \,, \tag{9}$$

we find for $g_{YM}^2 N_c \gg 1$

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 + \frac{135}{8} \, \zeta(3) \, (2 g_{YM}^2 N_c)^{-3/2} + \cdots \right) \,. \tag{10}$$

On the other hand, perturbative field theory calculations at weak coupling $g \ll 1$ give (see, *e.g.*, [30] and references therein)

$$\frac{\eta}{s} \sim \frac{1}{g^4 \log 1/g^2} \,. \tag{11}$$

The dependence of η/s on 't Hooft coupling in $\mathcal{N} = 4$ $SU(N_c)$ SYM (at infinite N_c) is shown schematically in fig. 1.

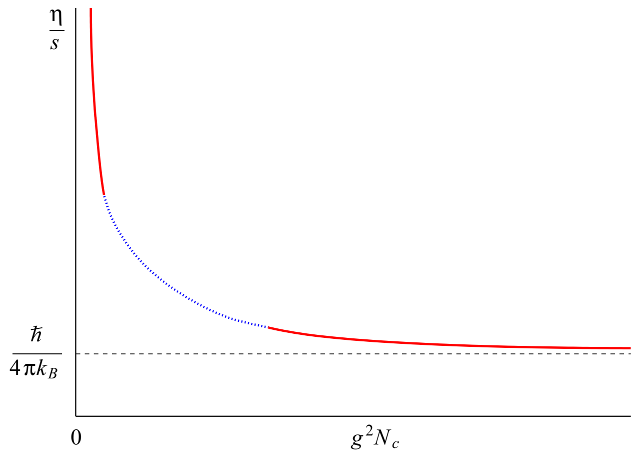

Fig. 1. The dependence of the ratio η/s on the 't Hooft coupling $g^2 N_c$ in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The ratio diverges in the limit $g^2 N_c \to 0$ and approaches $\hbar/4\pi k_B$ from above as $g^2 N_c \to \infty$. The ratio is unknown in the regime of intermediate 't Hooft coupling $g^2 N_c \sim 1$, but is likely to be a monotonic function of $g^2 N_c$. (Adapted from ref. [22].)

2.2 Transport coefficients from hydrodynamic poles of the retarded correlators

Hydrodynamics predicts that the retarded two-point correlation functions involving densities of conserved quantities must have singularities whose dispersion relation satisfies $\omega(q) \to 0$ as $q \to 0$, where q is the magnitude of the spatial momentum. For example, correlators of the components of the stress-energy tensor associated with the transverse fluctuation of the momentum density should exhibit a diffusion pole in the hydrodynamic regime,

$$G_{tx,xz}(\omega, \boldsymbol{q}) \sim -\frac{N_c^2 \pi T^3 \omega q}{8(i\omega - \mathcal{D} q^2)} \,, \tag{12}$$

where the corresponding dispersion relation

$$\omega(q) = -i \, \mathcal{D} \, q^2 + \mathrm{O}(q^3) \tag{13}$$

involves the diffusion constant $\mathcal{D} = \eta/(\varepsilon + P)$ which depends on the shear viscosity as well as on the energy density ε and the pressure P. (For homogeneous systems with zero chemical potential $\varepsilon + P = sT$.)

Similarly, correlators of the diagonal components of the stress-energy tensor corresponding to the propagation of the sound waves in a Yang-Mills plasma have a hydrodynamic pole with the dispersion relation

$$\omega(q) = v_s \, q - i \frac{\Gamma}{2} \, q^2 + \mathrm{O}(q^3) \,, \tag{14}$$

where $v_s = (\partial P / \partial \varepsilon)^{1/2}$ is the speed of sound. The sound attenuation constant Γ depends on shear and bulk viscosities η and ζ:

$$\Gamma = \frac{1}{\varepsilon + P} \left(\zeta + \frac{4}{3} \eta \right) \,.$$

Correlators $G_{\mu\nu,\sigma\rho}$ (as well as the correlators of the R-currents) in $\mathcal{N} = 4$ SYM (at infinite N_c and infinite 't Hooft coupling) can be computed analytically in the hydrodynamic approximation by solving the corresponding system of differential equations obeyed by the dual gravitational fluctuations. The functional form of the correlators coincides with the one expected from hydrodynamics. From the dispersion relations such as the ones given by eqs. (13), (14) we identify the transport coefficients of $\mathcal{N} = 4$ SYM in the above-mentioned limit [17,18]:

$$\eta = \frac{\pi}{8} N_c^2 T^3, \quad \zeta = 0, \quad v_s = \frac{1}{\sqrt{3}}, \quad D_R = \frac{1}{2\pi T}, \qquad (15)$$

where D_R is the R-charge diffusion constant.

The procedure of extracting the transport coefficients from the poles of the retarded correlators can be generalized to include any gravity dual [25]. At finite temperature, a generic stress-energy tensor two-point function depends (up to an index structure) on five scalar functions. On the gravity side, one can identify five gauge-invariant combinations of the metric perturbations that obey a coupled system of differential equations. The dispersion relations (13), (14) appear as the lowest eigenfrequencies of this system (the so-called quasinormal frequencies). Finding the quasinormal spectrum is a well posed (although often difficult) problem in mathematical physics. Examples of application of this general formalism to non-conformal theories can be found in [24,26]. In particular, in [26] for the strongly coupled $\mathcal{N} = 2^*$ SYM it was found that while the ratio η/s remains equal to $1/4\pi$, the bulk viscosity is non-zero and is proportional to the deviation of the speed of sound squared from its value in conformal theory,

$$\frac{\zeta}{\eta} \simeq \kappa \left(v_s^2 - \frac{1}{3} \right), \qquad (16)$$

where κ is a coefficient of order one. Transport properties of strongly coupled gauge theories with a spontaneously generated mass scale (the so called cascading gauge theories) were recently studied in [31].

2.3 The "membrane paradigm" approach

For a gravity dual with a metric of the form

$$\mathrm{d}s^2 = G_{00}(r)\,\mathrm{d}t^2 + G_{rr}(r)\,\mathrm{d}r^2 + G_{xx}(r) \sum_{i=1}^{p} (\mathrm{d}x^i)^2 + Z(r)\, K_{mn}(y)\, \mathrm{d}y^m \mathrm{d}y^n, \qquad (17)$$

where the components $G_{00}(r)$, $G_{rr}(r)$, $G_{xx}(r)$, and the "warping factor" $Z(r)$ depend only on the radial coordinate r, one can derive a generic formula for a diffusion coefficient [21]

$$D = \frac{\sqrt{-G(r_0)}Z(r_0)}{G_{xx}(r_0)\sqrt{-G_{00}(r_0)\,G_{rr}(r_0)}} \int_{r_0}^{\infty} \mathrm{d}r \frac{-G_{00}(r)G_{rr}(r)}{\sqrt{-G(r)}Z(r)} \qquad (18)$$

in terms of the metric components computed at the horizon $r = r_0$. Similarly, for the shear viscosity one has

$$\eta = s\,T \frac{\sqrt{-G(r_0)}}{\sqrt{-G_{00}(r_0)\,G_{rr}(r_0)}} \int_{r_0}^{\infty} \mathrm{d}r\, \frac{-G_{00}(r)\,G_{rr}(r)}{G_{xx}(r)\sqrt{-G(r)}}\,. \qquad (19)$$

These formulas were derived in the old, "pre-AdS/CFT" framework. However, it can be shown [32] that the same results arise from the computation of the lowest quasinormal frequency in the background (17).

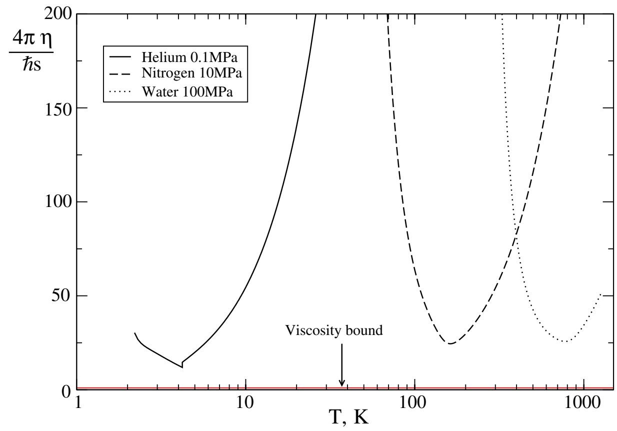

Fig. 2. The viscosity-entropy ratio for some common substances: helium, nitrogen and water. The ratio is always substantially larger than its value in theories with gravity duals, represented by the horizontal line marked "viscosity bound". (Adapted from ref. [22].)

3 Universality of shear viscosity in (super)gravity approximation

Calculations based on the Green-Kubo formula [22], the shear mode pole in the stress-energy tensor correlators [33] or the generic formula (19) [34] all reveal a rather surprising result that the ratio of the shear viscosity to the volume entropy density is universal and equal to $1/4\pi$ for *any* gauge theory in the regime of coupling and other parameters described by a *gravity* dual. Note that this statement says nothing about the behavior of the viscosity beyond the dual gravity approximation. (For example, for $\mathcal{N} = 4$ SYM one expects corrections to the ratio to appear for finite N_c and $g_{YN}^2 N_c$, as illustrated by eq. (10). Such corrections are not expected to be universal.)

The regime described by gravity duals is normally associated with an infinitely strong coupling. At the same time, at weak coupling the ratio η/s is typically very large. Restoring fundamental constants, we see that $\eta/s = \hbar/4\pi k_B \approx 6.08 \cdot 10^{-13}\,\mathrm{K \cdot s}$ is independent of the speed of light or the gravitational constant. One may put forward a *speculation* that perhaps the limit $1/4\pi$ serves as a universal lower bound on the value of η/s [21]. Arguments in favor of this speculation can be found in [21,22]. Experimental data for η/s for some common liquids are shown in fig. 2.

4 Conclusions

We have outlined a general approach for computing the transport coefficients of strongly coupled gauge theories from gravity. The ratio of shear viscosity to entropy density in the regime described by a gravity dual is universal and equal to $1/4\pi$ which is significantly lower than the corresponding perturbative result. Hydrodynamic models used to describe the elliptic flows observed at RHIC seem to suggest [35,36] that the ratio η/s in the non-perturbative regime of QCD should be small and relatively close to $1/4\pi$. Whether or not this intriguing observation is indeed related to the universality result for gravity duals remains to be seen.

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